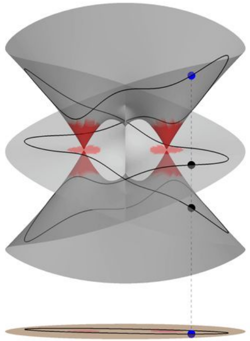
Galois/monodromy groups in 3D Reconstruction



# views	6	5	5	5	4	4	4	4	4	4
Configuration										
# solutions	$\approx 10^6$	11296	26240	11008	3040	4512	1728	32	544	544

# views	3	3	3	3	3	3	3	3	3	3
Configuration										
# solutions	360	552	480	264	432	328	480	240	64	216

# views	3	3	3	3	3	3	3	2	2	2
Configuration										
# solutions	312	224	40	144	144	144	64	20	16	12

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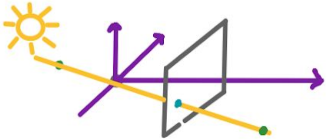
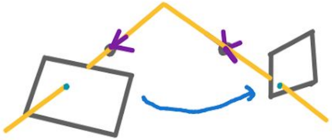
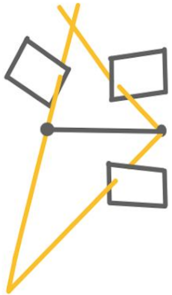
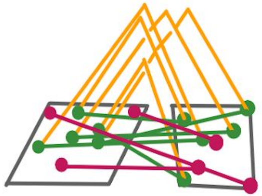




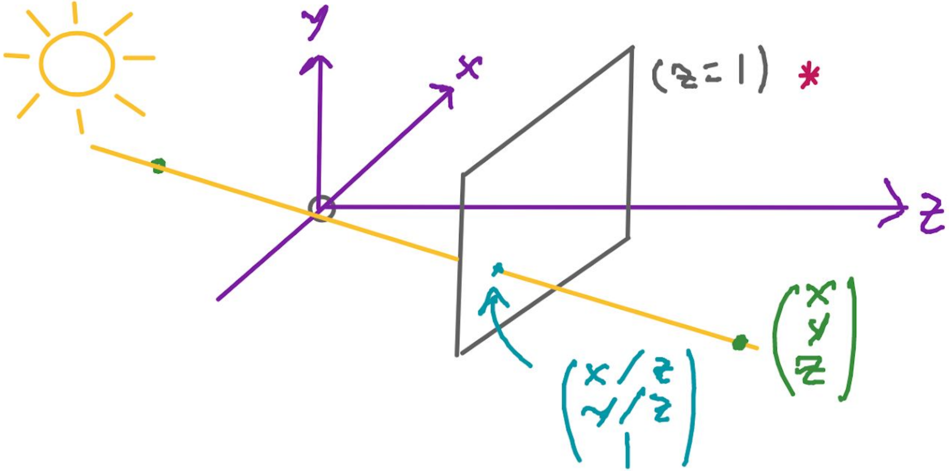


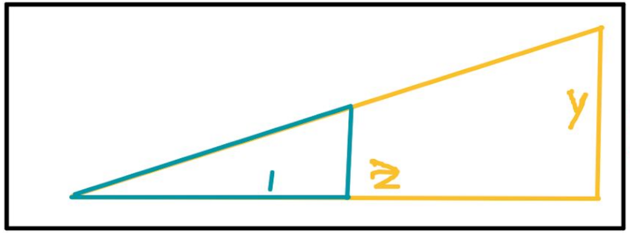


Overview

- Minimal problems are geometrically interesting enumerative problems / parametrized polynomial systems.
- Galois / monodromy / $\mathbb{C}$ groups identify structure in certain problems. Can be computed numerically.
- We consider classical & new examples of practical interest.

§1. Cameras & relative pose



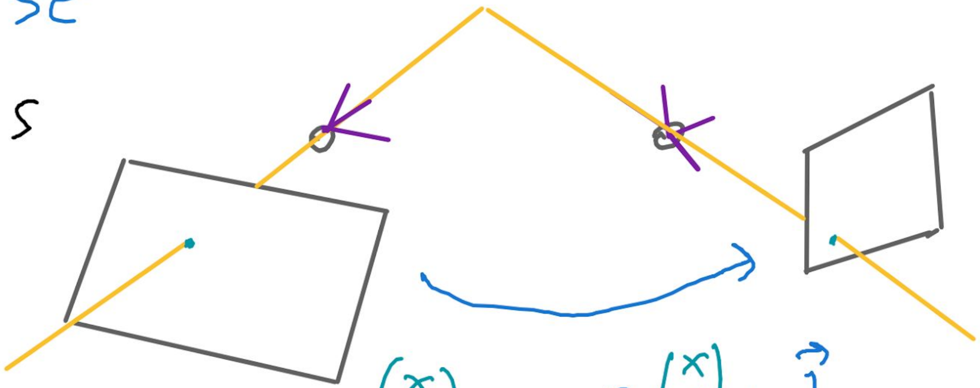


* Our cameras are calibrated

$$\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$$

$$\begin{pmatrix}x\\y\\z\\w\end{pmatrix} \mapsto \begin{pmatrix}x/z\\y/z\\1\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} = \underbrace{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}}_{\text{camera matrix}}\begin{pmatrix}x\\y\\z\\w\end{pmatrix}$$

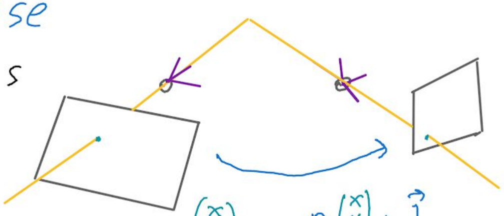
The relative pose $(R \,\vdots\, \vec{t}\,) \in SE(3)$ maps points between two camera frames.



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto R \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \vec{t}$$

Problem: Recover relative pose from corresponding features (e.g. points) in several images.

The relative pose $(R, \vec{t}) \in SE(3)$ maps points between two camera frames.



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto R \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \vec{t}$$

Problem: Recover relative pose from corresponding features (e.g. points) in several images.

Inherent ambiguity:

- Can only re over $\vec{t}$ up to scale.
- Relative pose determines world points (up to scale.)

Point correspondence:

$$\boxed{d_2 \vec{y} = d_1 R \vec{x} + \vec{t}}$$

Essential matrix (Longuet-Higgins, '81)

$$E = \underbrace{\begin{pmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{pmatrix}}_{[t]_\times} R$$

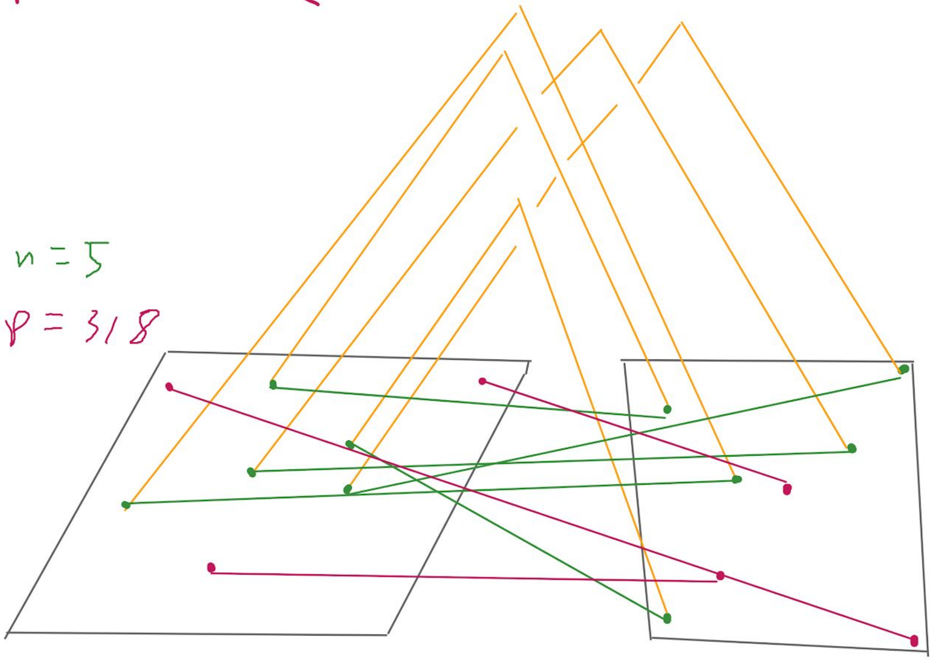
$$\boxed{\vec{y}^T E \vec{x} = 0}$$

n = # points used

p = Prob(outlier correspondence)

$n = 5$

$p = 3/8$



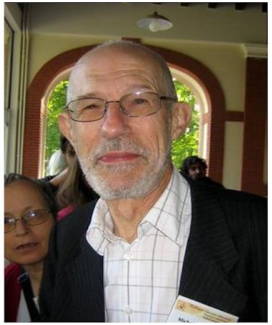
RANSAC (Fischer-Bolles '81)

needs $\approx 1/(1-p)^n$

iterations on average.

- n should be small
- Minimal solvers must be fast ($\approx$ 10-100 μs.)

$$V_{ess} = \mathrm{im}\left(\begin{array}{c} \varphi : SO(3,\mathbb{C}) \times \mathbb{P}(\mathbb{C}^3) \to \mathbb{P}(\mathbb{C}^{3\times 3}) \\ (R, \vec{t}) \mapsto [\vec{t}]_\times R \end{array} \right)$$



Algebraic geometry perspective (Demazure '88)

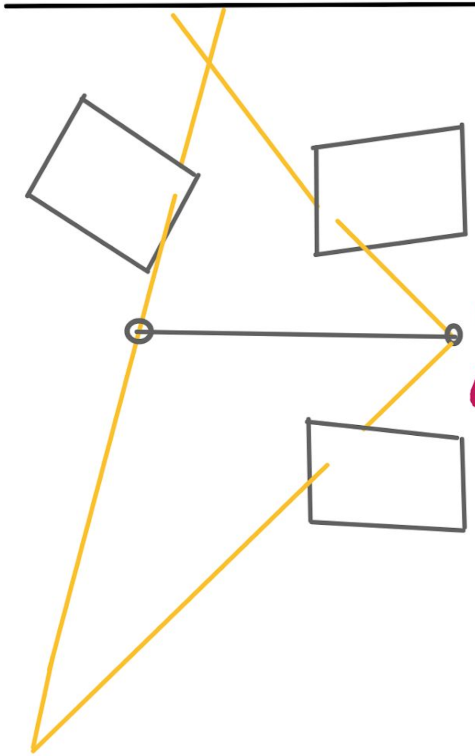
- $\dim V_{ess} = 5$ • $\deg V_{ess} = 10$

- $\mathcal{I}_{V_{ess}} = \langle EE^TE - \frac{1}{2}\mathrm{tr}(EE^T)E, \det E \rangle$
- φ is generically 2-1
- twisted pair $(R, \vec{t}) \mapsto \left(\left(2\frac{\vec{t}\,\vec{t}^T}{\vec{t}^T\vec{t}} - I \right) R, \vec{t} \right)$

§2 Minimal Problems as Branched Covers

$B = (\mathbb{P}^2 \times \mathbb{P}^2)^5$

$$X = \left\{ (X_1, \dots, X_5, R, \vec{t}, (\vec{x}_i, \vec{y}_i)) \in (\mathbb{P}^3)^5 \times SO(3) \times \mathbb{P}^2 \times B \;\middle|\; \begin{matrix} (I \;\; \vec{0})\vec{X}_i = \vec{x}_i \\ (R \;\; \vec{t})\vec{X}_i = \vec{y}_i \end{matrix} \right\}$$



d

- $\pi : X \twoheadrightarrow B$ is a branched cover of degree 20.
- Twisted pair $d : X \setminus \mathrm{ram}(\pi) \to X \setminus \mathrm{ram}(\pi)$ is a (rational) deck transformation.

$\Longrightarrow$ Galois/monodromy group acts imprimitely on $\pi^{-1}(b)$

Let $\pi : X \dashrightarrow B$ be a branched cover of degree e.

$X, B / \mathbb{C}$, irr. quasip.

$$\left(\exists \text{ a (rational) deck transformation } d : X \setminus \mathrm{ram}(\pi) \to X \setminus \mathrm{ram}(\pi) \iff \mathrm{Centralizer}(\mathrm{Gal}(\pi) \hookrightarrow S_e) \neq (\mathrm{id}) \right)$$

$$\Longrightarrow \left(\text{Monodromy group acts imprimitively} \iff \pi = \pi_2 \circ \pi_1 \text{ w/ } \deg(\pi_i) < e. \right)$$

5pt problem: $X \xrightarrow{\pi_1} V_{ess} \times B \xrightarrow{\pi_2} B$, $X \to B$

$E = [\vec{t}]_\times R \in V_{ess}, R$

$\Longrightarrow E = U \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} V^T$ (SVD)

$\Longrightarrow$

$R = U \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} V^T$

or $U \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T$

$[n] = \{1, \ldots, n\}$ S_n = symmetric group on $[n]$.

$S_n \wr_{[m]} S_m \hookrightarrow S_{mn}$ preserves partition into columns.

1	2	$\cdots$	m
$m+1$	$m+2$	$\cdots$	$2m$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$mn-m+1$	$mn-m+2$	$\cdots$	mn

⚠ Important to keep track of the action.

Minimal zoo

Want problems w/ simple features and small degree (Possibly after reformulation.)

# views	6	5	5	5	4	4	4	4	4	4
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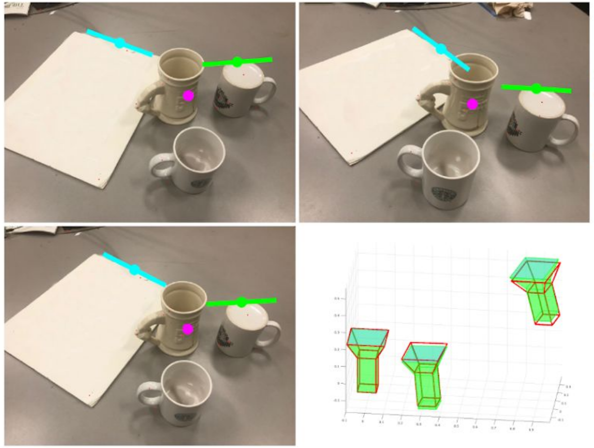
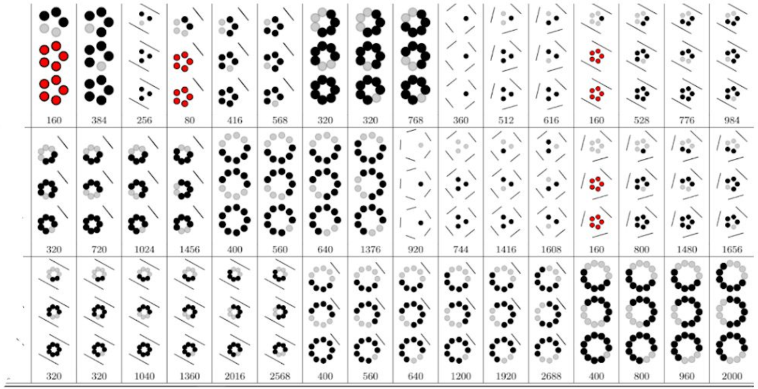
(D., Kohn, Leykin, Pajdla)



(Fabbri, D., Fan, Regan, ...)

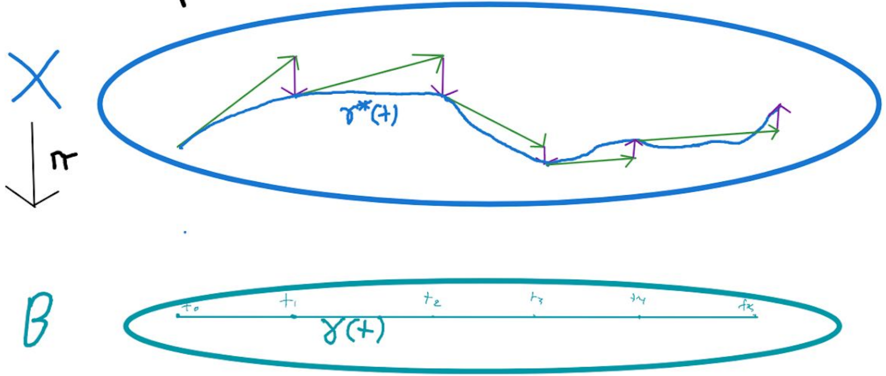


Thanks to ICERM

Let $\pi: X \twoheadrightarrow B$ be a branched cover of degree e.

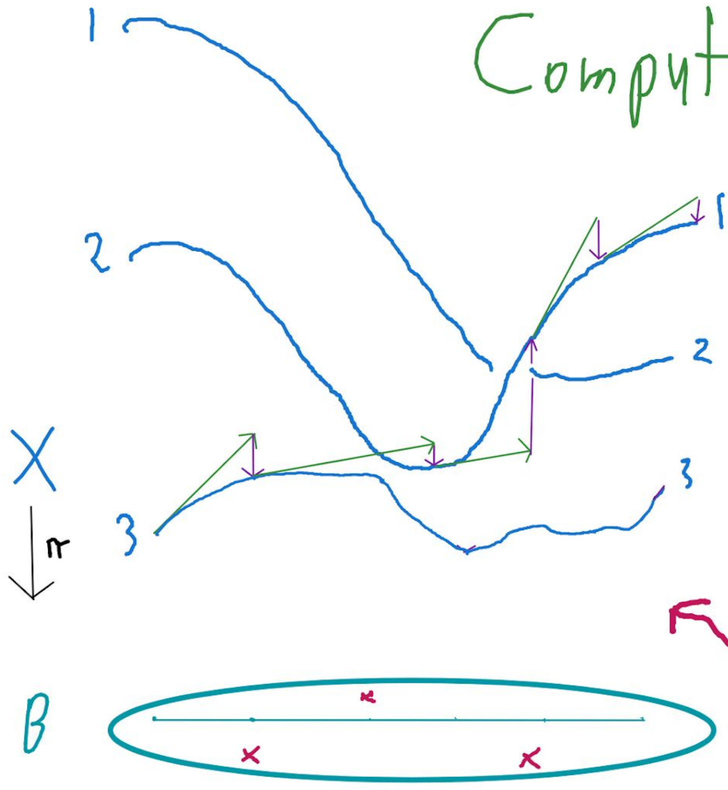
$\left(\begin{array}{l} \exists \text{ a (rational) deck transformation } d: X \setminus \mathrm{ram}(\pi) \to X \setminus \mathrm{ram}(\pi) \\ \Longleftrightarrow \mathrm{Centralizer}(\mathrm{Gal}(\pi) \hookrightarrow S_e) \neq (\mathrm{id}) \end{array} \right)$

$\Longrightarrow \left(\begin{array}{l} \text{Monodromy group acts imprimitively} \\ \Longleftrightarrow \pi = \pi_2 \circ \pi_1 \text{ w/ } \deg(\pi_i) < e. \end{array} \right)$



- Gröbner basis solvers typically compute eigenvectors of a $e \times e$ action matrix
- Homotopy continuation tracks e implicitly-defined lifts of path $\gamma: [0,1] \to B$ via predictor/corrector scheme

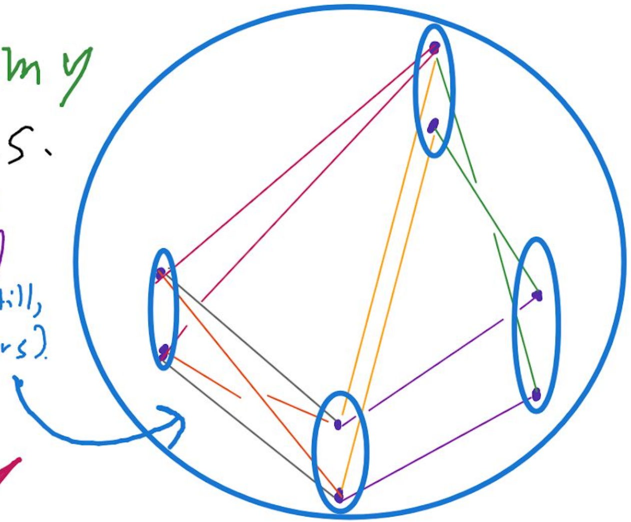
Computing monodromy



- Heuristics v.s.

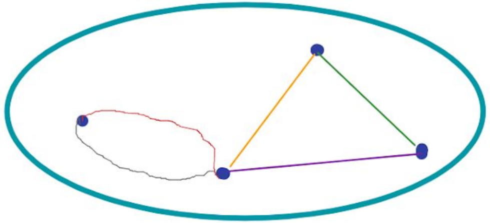
branch point method (Hauenstein, Rodriguez, Sottile)

- Homotopy graphs (D., Hill, Jensen, Lee, Leykin, Sommars).



Uncertainty

- Floating point, rounding errors
- Path-jumping
- Whole group or subgroup?

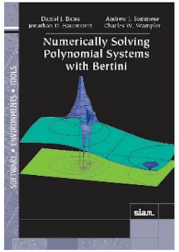


§3 Results & outlook

Our starting point was 5-point relative pose

$\mathrm{Gal}(\pi) \cong S_2 \wr S_{10} \cap A_{20}$

Computed w/ both

Bertini 

& MonodromySolver (a Macaulay2 package) 

$\mathrm{tr}(E^T \vec{x}_i \vec{y}_i^T) = \vec{y}_i^T E \vec{x}_i = 0$

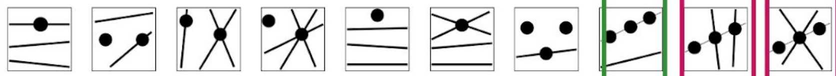
$(i = 1, \dots, 5.)$

A generic 5-secant $S = \langle \vec{x}_1 \vec{y}_1^T, \dots, \vec{x}_5 \vec{y}_5^T \rangle$ in the Segre variefy $\Sigma_{2,2} \hookrightarrow \mathbb{P}(\mathbb{C}^{3\times 3})$ satisfies $S \cap \Sigma_{2,2} = S$

$\Longrightarrow (\mathbb{P}^2 \times \mathbb{P}^2)^5 \dashrightarrow Gr(\mathbb{P}^3, \mathbb{P}^8)$ is dominant, gen. finite

$\Longrightarrow$ (almost) $\mathrm{Gal}\begin{pmatrix} V_{ess} \times B \\ \to B \end{pmatrix} \cong S_{10}$ (unif. pos. lemma)

Minimal problems with complete visibility (cal. relative pose.)

# views	6	5	5	5	4	4	4	4	4	4
Configuration										
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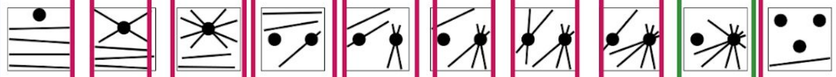
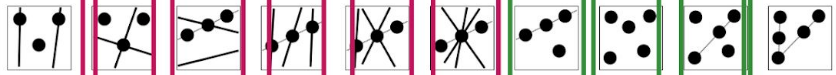
Configuration										
# solutions	360	552	480	264	432	328	480	240	64	216

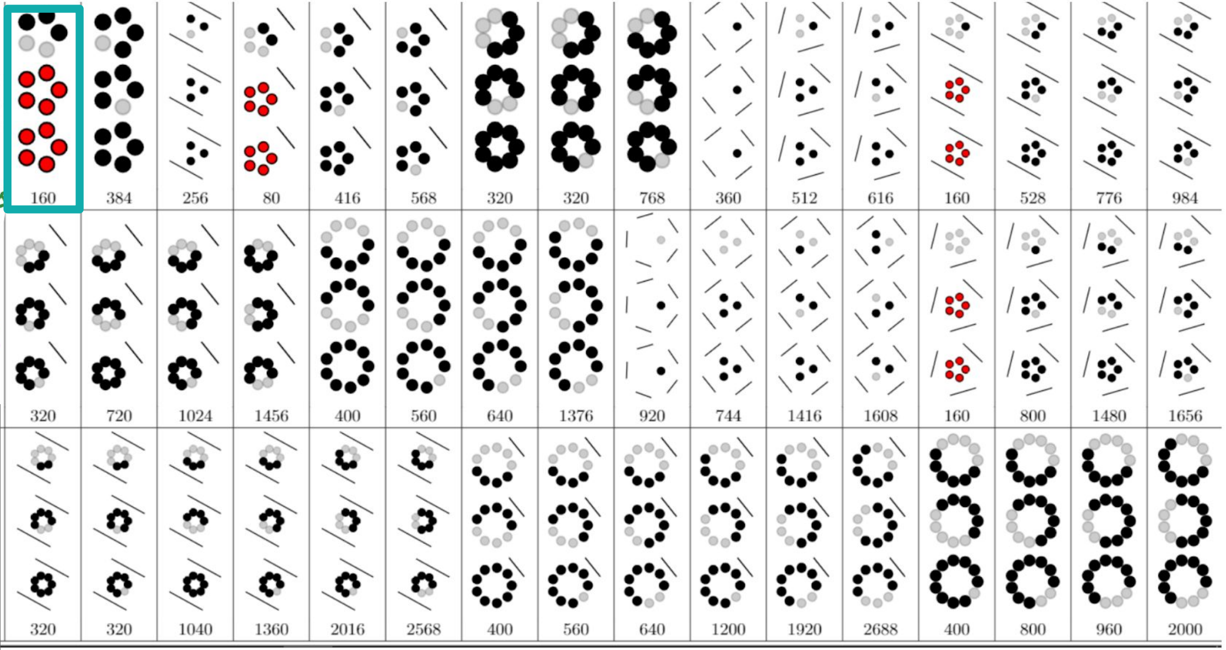
# views	3	3	3	3	3	3	3	2	2	2
Configuration										
# solutions	312	224	40	144	144	144	64	20	16	12

- Among problems of degree < 1,000, either full-symmetric or imprimitive.
- Factorizations for imprimitive cases always induced by deck transformations.

Partial visibility

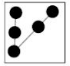
$$S_2 \wr \Big(\underbrace{S_4 \wr \big(S_2 \wr S_{10} \cap A_{20} \big)}_{\text{trivial centralizer in } S_{80}} \Big) \cap A_{160}$$

- Can have composite problems which are minimal.
- Not all factorizations are induced by deck transformations.



Five points in special position

2



12

Factorization recovers classical calibrated homography problem.

Galois Group:

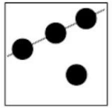
$$S_2 \wr \underbrace{(S_2 \wr S_3 \cap A_6)}_{\cong S_4 \text{ acting on pairs } 1 \leq i < j \leq 4} \cap A_{12}$$

```
GAP 4.10.2 of 19-Jun-2019
https://www.gap-system.org
Architecture: x86_64-pc-linux-gnu-default64-kv3
Configuration:  gmp 6.2.0, readline
Loading the library and packages ...
Packages:   Alnuth 3.1.1, AtlasRep 1.5.1, AutPGrp 1.10, CTblLib 1.2.2, FactInt 1.6.3,
            GAPDoc 1.6.2, IO 4.7.0, Polycyclic 2.14, PrimGrp 3.3.2, SmallGrp 1.3, TomLib 1.2.8,
            TransGrp 2.0.4
Try '??help' for help. See also '?copyright', '?cite' and '?authors'
gap> p0:= PermList([11, 12, 3, 4, 9, 10, 7, 8, 5, 6, 1, 2]);
p1:= PermList([12, 11, 7, 8, 10, 9, 3, 4, 2, 1, 6, 5]);
p2:= PermList([5, 6, 7, 8, 1, 2, 3, 4, 9, 10, 11, 12]);
p3:= PermList([6, 5, 11, 12, 2, 1, 9, 10, 4, 3, 8, 7]);
p4:= PermList([11, 12, 7, 8, 9, 10, 3, 4, 1, 2, 5, 6]);
p5:= PermList([6, 5, 10, 9, 2, 1, 12, 11, 7, 8, 3, 4]);
p6:= PermList([5, 6, 11, 12, 1, 2, 9, 10, 3, 4, 7, 8]);
G:=Group(p0, p1, p2, p3, p4, p5, p6);
(1,11)(2,12)(5,9)(6,10)
gap> (1,12,5,10)(2,11,6,9)(3,7)(4,8)
gap> (1,5)(2,6)(3,7)(4,8)
gap> (1,6)(2,5)(3,11,8,10)(4,12,7,9)
gap> (1,11,5,9)(2,12,6,10)(3,7)(4,8)
gap> (1,6)(2,5)(3,10,8,11)(4,9,7,12)
gap> (1,5)(2,6)(3,11,7,9)(4,12,8,10)
gap> Group([ (1,11)(2,12)(5,9)(6,10), (1,12,5,10)(2,11,6,9)(3,7)(4,8), (1,5)(2,6)(3,7)(4,8), (1,6)(2,5)
(3,11,8,10)(4,12,7,9), (1,11,5,9)(2,12,6,10)(3,7)(4,8), (1,6)(2,5)(3,10,8,11)(4,9,7,12), (1,5)(2,6)
(3,11,7,9)(4,12,8,10) ])
gap> G1 := Image(ActionHomomorphism(G, Blocks(G, [1..12]), OnSets));
Group([ (1,6)(3,5), (1,6,3,5)(2,4), (1,3)(2,4), (1,3)(2,6,4,5), (1,6,3,5)(2,4), (1,3)(2,5,4,6), (1,3)(2,6,4,5) ])
gap> StructureDescription(G1);
"S4"
gap> StructureDescription(G);
"(C2 x C2) : S4"
gap> G2 := Image(ActionHomomorphism(G1, Blocks(G1, [1..6]), OnSets));
Group([ (1,3,2), (2,3) ])
```

Promising new candidates

(3  64 probably too exotic)

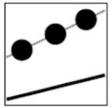
3



64

$$S_2 \wr \left(S_2 \wr S_{\underline{16}} \cap A_{32}\right) \cap A_{64}$$

4



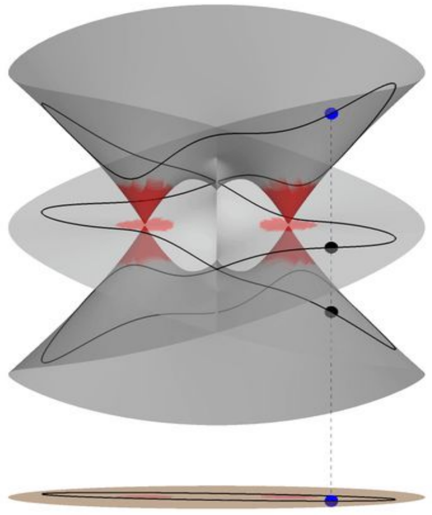
32

$$S_2 \wr \left(S_2 \wr \left(S_2 \wr \left(S_2 \wr S_{\underline{2}} \cap A_4\right)\right)\right)$$

+ explicit deck transformation)

Outlook

- So far, we see the following patterns:
  - Full-symmetric/imprimitive dichotomy
  - All imprimitive/decomposable minimal problems are induced by deck transformation (complete visibility) or elimination/projection (partial visibility)



Thanks for your attention!

- Present and future work
  - Find factorizations & build faster solvers
  - Expand scope to other camera models & beyond relative pose